Meeting 10 — Math 22

Reading Debrief

- Discuss Activity 10.7.2 w/ your group.
- Questions from Sections 10.7.1 - 10.7.2?

Section 10.7.2 — Second-Derivative Test

The Second Derivative Test.

Suppose (x_0, y_0) is a critical point of the function f for which $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Let D be the quantity defined by

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2.$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
2. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
3. If $D < 0$, then f has a saddle point at (x_0, y_0).
4. If $D = 0$, then this test yields no information about what happens at (x_0, y_0).

The quantity D is called the *discriminant* of the function f at (x_0, y_0).

The discriminant D is just the determinant of the Hessian matrix

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

The values of $f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$ at the point (x_0, y_0) determine a (Taylor) polynomial of degree 2 that approximates the function f at (x_0, y_0). The Second-Derivative test gives conditions under which the Taylor polynomial has local max/min or saddle.

Case 1 $\rightarrow$ Case 3

Case 2 $\rightarrow$ Case 4 Anything can happen

Activity 10.7.4

- Complete w/ your group.
- Class discussion.

(a) $\nabla f(x,y) = \langle 9x^2 - 9, 2y + 4 \rangle = \langle 0, 0 \rangle$

Two critical points $(1,-2), (-1,-2)$.

$$D(x,y) = \det \begin{bmatrix} f_{xx} = 18x & f_{xy} = 0 \\ f_{yx} = 0 & f_{yy} = 2 \end{bmatrix} = 36x$$

$D(1,-2) = 36 > 0$ $\qquad$ $D(-1,-2) = -36 < 0$

$f_{xx}(1,-2) = 18 > 0$ $\qquad$ $(-1,2)$ is a saddle point

$(1,-2)$ Local min

(b) $\nabla f(x,y) = \langle y - \frac{2}{x^2}, x - \frac{4}{y^2} \rangle = 0$

$y = \frac{2}{x^2} \quad x = \frac{4}{y^2} \Rightarrow x = \frac{4}{(\frac{2}{x^2})^2} = x^4$

$\Rightarrow x = 1, \ y = 2$

There is one critical point $(1,2)$

$f_{xx} = \frac{4}{x^3} \qquad f_{xy} = 1$

$f_{yx} = 1 \qquad f_{yy} = \frac{8}{y^3}$

$$D(1,2) = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} = 3 > 0$$

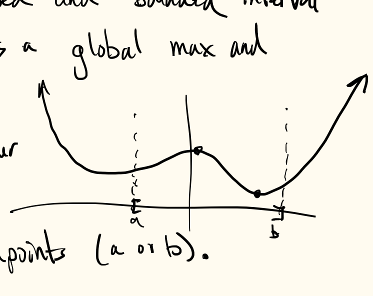
$f_{xx}(1,2) = 4 > 0$ $\qquad$ $(1,2)$ is a local min

Activity 10.7.5

- Complete and discuss w/ your group.
- Class discussion.

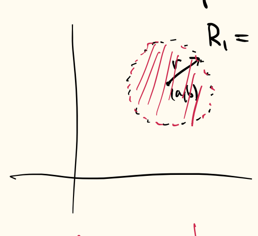
Section 10.7.3 — Optimization on a Restricted Domain

Extreme Value Thm (Calc I) Suppose $f(x)$ is continuous on a closed and bounded interval $[a,b]$. Then $f(x)$ has a global max and a global min.
The absolute extrema occur either at critical points ($f'(x) = 0$) or at the endpoints (a or b).

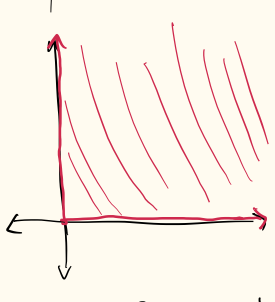


In multivariable calc, we need to consider closed and bounded regions in the xy-plane instead of closed intervals.

The meaning of "bounded region" should be clear. For example:



$R_1 = \{(x,y) \mid (x-a)^2 + (y-b)^2 < r\}$
open disk
Bounded



$R_2 = \{(x,y) \mid x, y \geq 0\}$
Not Bounded

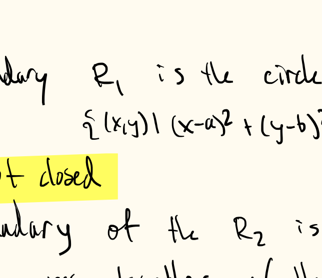
A region R is closed if it contains its boundary.
For example
- The boundary R_1 is the circle $\{(x,y) \mid (x-a)^2 + (y-b)^2 = r\}$

R_1 is not closed

- The boundary of the R_2 is the positive x and y axes together w/ the origin.

R_2 is closed

We are interested in closed and bounded.



The region $R_3 = \{(x,y) \mid (x-a)^2 + (y-b)^2 \leq r\}$ is closed and bounded.

Extreme Value Theorem Let $f(x,y)$ be a continuous function on a closed and bounded region R. Then f has an absolute max/min on R.

Example 10.7.8. Suppose the temperature T at each point on the circular plate $x^2 + y^2 \leq 1$ is given by

$$T(x,y) = 2x^2 + y^2 - y.$$

The domain $R = \{(x,y) : x^2 + y^2 \leq 1\}$ is a closed and bounded region, as shown on the left of Figure 10.7.9, so the Extreme Value Theorem assures us that T has an absolute maximum and minimum on the plate. The graph of T over its domain R is shown in Figure 10.7.9. We will find the hottest and coldest points on the plate.

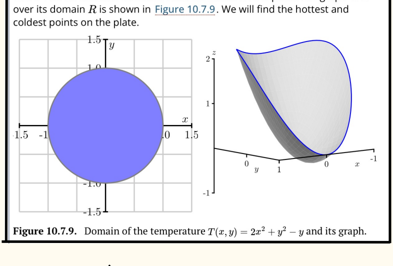


Figure 10.7.9. Domain of the temperature $T(x,y) = 2x^2 + y^2 - y$ and its graph.

Find critical points in the region:
$0 = \nabla T = \langle 4x, 2y - 1 \rangle$
One critical point $(0, \frac{1}{2})$ inside R.

Find critical points on the boundary:
- Parametrize the boundary of R

$x(t) = \cos t \quad y(t) = \sin t \quad 0 \leq t \leq 2\pi$

- Compose the function w/ the parametrization

$T(x(t), y(t))$

- Optimize $T(x(t), y(t))$ using calc I.

$$0 = \frac{d}{dt} T(x(t), y(t))$$
$$\underset{\text{(chain rule)}}{=} \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$
$$= -4\cos t \sin t + (2\sin t - 1)\cos t$$
$$= \cos t(-2\sin t - 1)$$

$\cos t = 0 \qquad \sin t = -\frac{1}{2}$

$t = \pi/2, 3\pi/2 \qquad t = 7\pi/6, 11\pi/6$

The corresponding points are

$(t = \pi/2) \ (0,1) \qquad (t = 7\pi/6) \ (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$

$(t = 3\pi/2) \ (0,-1) \qquad (t = 11\pi/6) \ (\frac{\sqrt{3}}{2}, -\frac{1}{2})$

There are 5 possible points where a global max/min can occur.
- Plug all points into the function and see which is largest/smallest.

$T(0, 1/2) = -1/4 \qquad T(0,1) = 0 \qquad T(-\frac{\sqrt{3}}{2}, -\frac{1}{2}) = \frac{9}{4}$

$T(\frac{\sqrt{3}}{2}, -\frac{1}{2}) = \frac{9}{4} \qquad T(0,-1) = 2$

The maximum value of $9/4$ occurs at $(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2})$, the minimum value of $-1/4$ occurs at $(0, 1/2)$.

Activity 10.7.6

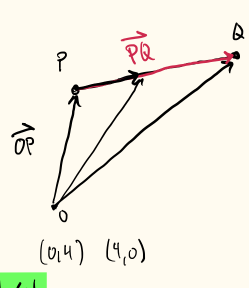
- Complete w/ your group.
- Class discussion.

(a) $\nabla f(x,y) = \langle 2x - 4, -6y + 6 \rangle = 0$

There is one critical point $(2,1)$ inside the triangle.

(b) $r(t) = \vec{OP} + t\vec{PQ}$

$= (0,0) + t(4,0)$



$x_1(t) = 4t \quad y_1(t) = 0 \quad 0 \leq t \leq 1$

(c) $x_2(t) = 0 \quad y_2(t) = 4t \quad 0 \leq t \leq 1$

(d) $x_3(t) = 4 - 4t \quad y_3(t) = 4t \quad 0 \leq t \leq 1$

(e) $f(x_1, y_1) = 16t^2 - 16t \qquad 0 = 32t - 16$

$t = \frac{1}{2} \rightsquigarrow (2,0)$

$f(x_2, y_2) = -48t^2 + 24t \qquad 0 = -96t + 24$

$t = \frac{1}{4} \rightsquigarrow (0,1)$

$f(x_3, y_3) = (4 - 4t)^2 - 48t^2 - 16t + 24t$
$= 16 - 32t^2 - 48t^2 - 16t + 24t + 16t^2$
$= 16 - 64t^2 + 8t$

$0 = -128t + 8$

$t = \frac{1}{16} \rightsquigarrow (15/4, 1/4)$

Also have to check the corners: $(4,0)$ $(0,4)$, $(0,0)$

Evaluate $f(x,y)$ at all 7 points.

$f(0,0) = 0 \qquad f(4,0) = 0 \qquad f(0,4) = -24$

$f(2,0) = -4 \qquad f(0,1) = 3 \qquad f(2,1) = -1$

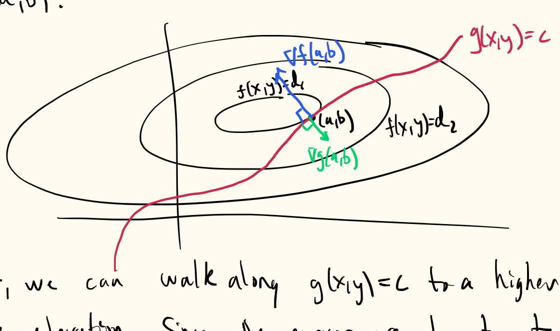
$f(15/4, 1/4) = \frac{3}{8}$

Section 10.8 — Lagrange Multipliers

Reading Debrief

- Look at Preview Activity 10.8.1 as a class.

Suppose we want to optimize a function $f(x,y)$ subject to a constraint equation $g(x,y) = c$. The constraint eq. is a level curve of g. If a point (a,b) which optimizes f and satisfies $g(a,b) = c$, then the level curve $f(x,y) = f(a,b)$ is tangent to the constraint curve at (a,b).



If not, we can walk along $g(x,y) = c$ to a higher or lower elevation. Since the curves are tangent at (a,b), the gradient $\nabla f(a,b)$ and $\nabla g(a,b)$ are parallel. There exists a scalar λ s.t

$$\nabla f = \lambda \nabla g.$$

This is the **Lagrange Multiplier Equation**. It provides a method for finding "critical points" of a function subject to a constraint.